

RECENT DEVELOPMENTS IN ALGEBRAIC COMBINATORICS

BY

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ABSTRACT

We survey three recent developments in algebraic combinatorics. The first is the theory of cluster algebras and the Laurent phenomenon of Sergey Fomin and Andrei Zelevinsky. The second is the construction of toric Schur functions and their application to computing three-point Gromov–Witten invariants, by Alexander Postnikov. The third development is the construction of intersection cohomology for nonrational fans by Paul Bressler and Valery Lunts and its application by Kalle Karu to the toric h -vector of a nonrational polytope. We also briefly discuss the “half hard Lefschetz theorem” of Ed Swartz and its application to matroid complexes.

1. Introduction

In a previous paper [32] we discussed three recent developments in algebraic combinatorics. In the present paper we consider three additional topics, namely, the Laurent phenomenon and its connection with Somos sequences and related sequences, the theory of toric Schur functions and its connection with the quantum cohomology of the Grassmannian and 3-point Gromov–Witten invariants, and the toric h -vector of a convex polytope.

Note: The notation \mathbb{C} , \mathbb{R} , and \mathbb{Z} denotes the sets of complex numbers, real numbers, and integers, respectively.

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2. The Laurent phenomenon

Consider the recurrence

$$(1) \quad a_{n-1}a_{n+1} = a_n^2 + (-1)^n, \quad n \geq 1,$$

with the initial conditions $a_0 = 0$, $a_1 = 1$. A priori it isn't evident that a_n is an integer for all n . However, it is easy to check (and is well-known) that a_n is given by the Fibonacci number F_n . The recurrence (1) can be "explained" by the fact that F_n is a linear combination of two exponential functions. Equivalently, the recurrence (1) follows from the addition law for the exponential function e^x or for the sine, viz.,

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

In the 1980's Michael Somos set out to do something similar involving the addition law for elliptic functions. Around 1982 he discovered a sequence, now known as Somos-6, defined by quadratic recurrences and seemingly integer valued [25]. A number of people generalized Somos-6 to Somos- N for any $N \geq 4$. The sequences Somos-4 through Somos-7 are defined as follows. (The definition of Somos- N should then be obvious.)

$$\begin{aligned} a_n a_{n-4} &= a_{n-1} a_{n-3} + a_{n-2}^2, \quad n \geq 4; \quad a_i = 1 \text{ for } 0 \leq i \leq 3, \\ a_n a_{n-5} &= a_{n-1} a_{n-4} + a_{n-2} a_{n-3}, \quad n \geq 5; \quad a_i = 1 \text{ for } 0 \leq i \leq 4, \\ a_n a_{n-6} &= a_{n-1} a_{n-5} + a_{n-2} a_{n-4} + a_{n-3}^2, \quad n \geq 6; \quad a_i = 1 \text{ for } 0 \leq i \leq 5, \\ a_n a_{n-7} &= a_{n-1} a_{n-6} + a_{n-2} a_{n-5} + a_{n-3} a_{n-4}, \quad n \geq 7; \quad a_i = 1 \text{ for } 0 \leq i \leq 6. \end{aligned}$$

It was conjectured that all four of these sequences are integral, i.e., all their terms are integers. Surprisingly, however, the terms of Somos-8 are not all integers. The first nonintegral value is $a_{17} = 420514/7$. Several proofs were quickly given that Somos-4 and Somos-5 are integral, and independently Hickerson and Stanley showed the integrality of Somos-6 using extensive computer calculations. Many other related sequences were either proved or conjectured to be integral. For example, Robinson conjectured that if $1 \leq p \leq q \leq r$ and $k = p + q + r$, then the sequence defined by

$$(2) \quad a_n a_{n-k} = a_{n-p} a_{n-k+p} + a_{n-q} a_{n-k+q} + a_{n-r} a_{n-k+r},$$

with initial conditions $a_i = 1$ for $0 \leq i \leq k-1$, is integral. A nice survey of the early history of Somos sequences, including an elegant proof by Bergman of the integrality of Somos-4 and Somos-5, was given by Gale [16].

A further direction in which Somos sequences can be generalized is the introduction of parameters. The coefficients of the terms of the recurrence can be generic (i.e., indeterminates), as first suggested by Gale, and the initial conditions can be generic. Thus, for instance, the generic version of Somos-4 is

$$(3) \quad a_n a_{n-4} = x a_{n-1} a_{n-3} + y a_{n-2}^2,$$

with initial conditions $a_0 = a$, $a_1 = b$, $a_2 = c$, and $a_3 = d$, where x, y, a, b, c, d are independent indeterminates.

Thus a_n is a rational function of the six indeterminates. A priori the denominator of a_n can be a complicated polynomial, but it turns out that when a_n is reduced to lowest terms the denominator is always a monomial, while the numerator is a polynomial with integer coefficients. In other words, $a_n \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}]$, the Laurent polynomial ring over \mathbb{Z} in the indeterminates x, y, a, b, c, d . This unexpected appearance of Laurent polynomials when more general rational functions are expected is called by Fomin and Zelevinsky [13] the **Laurent phenomenon**.

Until recently all work related to Somos sequences and the Laurent phenomenon was of an ad hoc nature. Special cases were proved by special techniques, and there was no general method for approaching these problems. This situation changed with the pioneering work of Fomin and Zelevinsky [12], [14], [3] on **cluster algebras**. These are a new class of commutative algebras originally developed in order to create an algebraic framework for dual-canonical bases and total positivity in semisimple groups. A cluster algebra is generated by the union of certain subsets, known as **clusters**, of its elements. Every element y of a cluster is a rational function $F_y(x_1, \dots, x_n)$ of the elements of any other cluster $\{x_1, \dots, x_n\}$. A crucial property of cluster algebras, not at all evident from their definition, is that $F_y(x_1, \dots, x_n)$ is in fact a **Laurent polynomial** in the x_i 's. Fomin and Zelevinsky realized that their proof of this fact could be modified to apply to a host of combinatorial conjectures and problems concerning integrality and Laurentness. Let us note that although cluster algebra techniques have led to tremendous advances in the understanding of the Laurent phenomena, they do not appear to be the end of the story. There are still many conjectures and open problems seemingly not amenable to cluster algebra techniques.

We will illustrate the technique of Fomin and Zelevinsky for the Somos-4 sequence. Consider Figure 1. Our variables will consist of x_0, x_1, x_2, \dots and $x_{2'}, x_{3'}, x_{4'}, \dots$. The figure shows part of an infinite tree T , extending to the right. (We have split the tree into two rows. The leftmost edge of the second

row is a continuation of the rightmost edge of the first row.) The tree consists of a **spine**, which is an infinite path drawn at the top, and two legs attached to each vertex of the spine except the first. The spine vertices $v = v_i$, $i \geq 0$, are drawn as circles with i inside. This stands for the set of variables (cluster) $C_v = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$. Each spine edge e has a numerical label a_e on the top left of the edge, and another b_e on the top right, as well as a polynomial label P_e above the middle of the edge. A leg edge e has a numerical label a_e at the top, a polynomial label P_e in the middle, and a label $b_e = a'_e$ at the bottom.

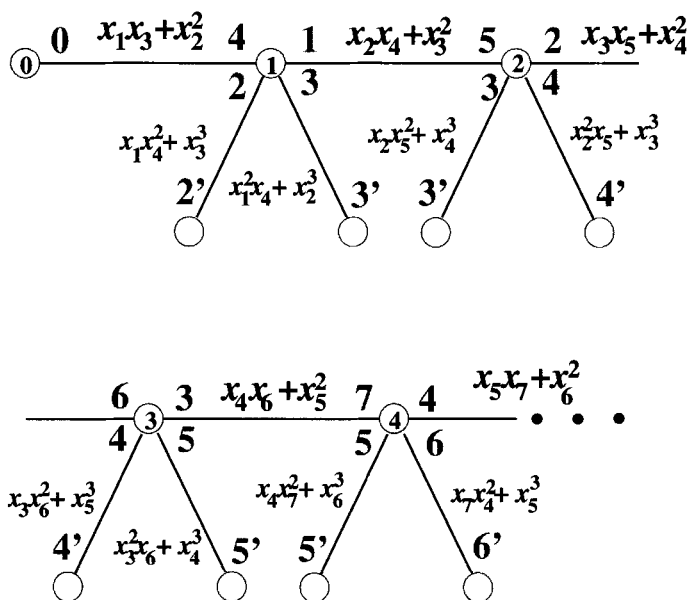


Figure 1. The Somos-4 cluster tree T .

Moreover, if e is incident to the spine vertex v and leg vertex w , then w has associated with it the cluster $C_w = (C_v \cup \{x_{a'_e}\}) - \{x_{a_e}\}$. Thus for any edge e , if the label a_e is next to vertex v and the label b_e is next to vertex w , then $C_w = (C_v \cup \{x_{b_e}\}) - \{x_{a_e}\}$.

Let e be an edge of T with labels a_e , b_e , and P_e . These labels indicate that the variables x_{a_e} and x_{b_e} are related by the formula

$$x_{a_e}x_{b_e} = P_e.$$

(In the situation of cluster algebras, this would be a relation satisfied by the

generators x_i .) For instance, the leftmost edge of T yields the relation

$$x_0x_4 = x_1x_3 + x_2^2.$$

In this way all variables x_i and x'_i become rational functions of the “initial cluster” $C_0 = \{x_0, x_1, x_2, x_3\}$.

The edge labels of T can be checked to satisfy the following four conditions:

- Every internal vertex v_i , $i \geq 1$, has the same degree, namely four, and the four edge labels “next to” v_i are $i, i+1, i+2, i+3$, the indices of the cluster variables associated to v_i .
- The polynomial P_e does not depend on x_{a_e} and x_{b_e} , and is not divisible by any variable x_i or x'_i .
- Write \bar{P}_e for P_e with each variable x_j and x'_j replaced with $x_{\bar{j}}$, where \bar{j} is the least positive residue of j modulo 4. If e and f are consecutive edges of T then the polynomials \bar{P}_e and $\bar{P}_{f,0} := \bar{P}_f|_{x_{\bar{a}_e}=0}$ are relatively prime elements of $\mathbb{Z}[x_1, x_2, x_3, x_4]$. For instance, the leftmost two top edges of T yield that $x_1x_3 + x_2^2$ and $(x_2x_4 + x_3^2)|_{x_4=0} = x_3^2$ are coprime.
- If e, f, g are three consecutive edges of T such that $\bar{a}_e = \bar{a}_g$, then

$$(4) \quad L \cdot \bar{P}_{f,0}^b \cdot \bar{P}_e = \bar{P}_g \mid_{x_{\bar{a}_f} \leftarrow \bar{P}_{f,0}/x_{\bar{a}_f}}$$

where L is a Laurent monomial, $b \geq 0$, and $x_{\bar{a}_f} \leftarrow \bar{P}_{f,0}/x_{\bar{a}_f}$ denotes the substitution of $\bar{P}_{f,0}/x_{\bar{a}_f}$ for $x_{\bar{a}_f}$. For instance, let e be the leftmost leg edge and f, g the second and third spine edges. Thus $\bar{a}_e = \bar{a}_g = 2$ and $\bar{a}_f = 1$. Equation (4) becomes

$$L \cdot (x_2x_4 + x_3^2)_{x_2=0}^b \cdot (x_1x_4^2 + x_3^3) = (x_1x_3 + x_2^2) \mid_{x_1 \leftarrow x_3^2/x_1},$$

which holds for $b = 0$ and $L = 1/x_1$, as desired.

The above properties may seem rather bizarre, but they are precisely what is needed to be able to prove by induction that every variable x_i and x'_i is a Laurent polynomial with integer coefficients in the initial cluster variables x_0, x_1, x_2, x_3 (or indeed in the variables of any cluster). We will not give the proof here, though it is entirely elementary. Of crucial importance is the periodic nature of the labelled tree T . Each edge is labelled by increasing all indices by one from the corresponding edge to its left. This means that the *a priori* infinitely many conditions that need to be checked are reduced to a (small) finite number.

It follows from the relations $x_ix_{i+4} = x_{i+1}x_{i+3} + x_{i+2}^2$ that x_n is just the n th term of Somos-4 with the generic initial conditions x_0, x_1, x_2, x_3 . Since x_n is a Laurent polynomial with integer coefficients in the variables x_0, x_1, x_2, x_3 , if

we set $x_0 = x_1 = x_2 = x_3 = 1$ then x_n becomes an integer. In this way the integrality of the original Somos-4 sequence is proved by Fomin and Zelevinsky.

By similar arguments Fomin and Zelevinsky prove a host of other integrality theorems, as mentioned above. In particular, they prove the integrality of Somos-5, Somos-6, Somos-7, and the Robinson recurrence (2) by this method. This gives the first proof of the integrality of Somos-7 (and the first published proof for Somos-6), as well as a proof of Robinson's conjecture. By a refinement of the argument of Fomin and Zelevinsky for Somos-4, David Speyer [26] has shown that for the generic Somos-4 sequence (3) (with generic initial conditions), and similarly for generic Somos-5, the coefficients of the Laurent polynomial x_n are *nonnegative*. Nonnegativity remains open for generic Somos-6 and Somos-7. The reader might find it instructive to modify (straightforwardly) the graph T of Figure 1 to prove the following [13, Example 3.3].

2.1 THEOREM: *Let a, b , and c be positive integers, and let the sequence y_0, y_1, \dots satisfy the recurrence*

$$y_k = \frac{y_{k-3}^a y_{k-1}^c + y_{k-2}^b}{y_{k-4}}.$$

Then each y_i is a Laurent polynomial with integer coefficients in the initial terms y_0, y_1, y_2, y_3 .

Once the integrality of a recurrence is proved, it is natural to ask for a combinatorial proof.

In the case of Somos-4, we would like to give a combinatorial interpretation to the terms a_n and from this a combinatorial proof of the recurrence

$$a_n a_{n-4} = a_{n-1} a_{n-3} + a_{n-2}^2.$$

A clue as to how this might be done comes from the observation that the rate of growth of a_n is roughly quadratically exponential. Indeed, the function α^{n^2} satisfies the Somos-4 recurrence if $\alpha^8 = \alpha^2 + 1$. A previously known enumeration problem whose solution grows quadratically exponentially arises from the theory of matchings or domino (dimer) tilings. Let G be a finite graph, which we assume for convenience has no loops (vertices connected to themselves by an edge). A **complete matching** of G consists of a set of vertex-disjoint edges that cover all the vertices. Thus G must have an even number $2m$ of vertices, and each

complete matching contains m edges.

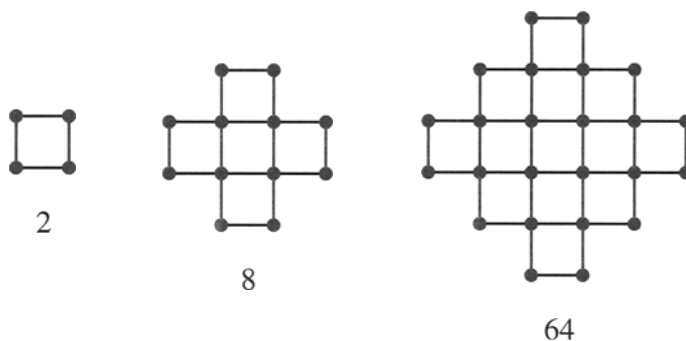


Figure 2. The Aztec diamond graphs AZ_n for $1 \leq n \leq 3$.

Figure 2 shows a sequence of graphs AZ_1, AZ_2, AZ_3, \dots , whose general definition should be clear from the figure. These graphs were introduced by Elkies, Kuperberg, Larsen and Propp [10], [11], who called them (essentially) **Aztec diamond graphs**. They give four proofs that the number of complete matchings of AZ_n is $2^{\binom{n+1}{2}}$. Since this number grows quadratically exponentially, Jim Propp got the idea that the terms a_n of Somos-4 might count the number of complete matchings in a planar graph S_n for which the Somos-4 recurrence could be proved combinatorially. The undergraduate research team REACH [24], directed by Propp, and independently Bousquet-Mélou, Propp and West [5] succeeded in finding such graphs S_n in the spring of 2002 [24]. Figure 3 shows the “Somos-4 graphs” S_4, S_5, S_6, S_7 along with their number of complete matchings.

3. Gromov–Witten invariants and toric Schur functions

Let Gr_{kn} denote the set of all k -dimensional subspaces of the n -dimensional complex vector space \mathbb{C}^n . We call Gr_{kn} the **Grassmann variety** or **Grassmannian**. It has the structure of a complex projective variety of dimension $k(n-k)$ and is naturally embedded in complex projective space $P^{\binom{n}{k}-1}(\mathbb{C})$ of dimension $\binom{n}{k} - 1$. The cohomology ring $H^*(\text{Gr}_{kn}) = H^*(\text{Gr}_{kn}; \mathbb{Z})$ is the fundamental object for the development of classical **Schubert calculus**, which is concerned, at the enumerative level, with counting the number of linear subspaces that satisfy certain geometric conditions. For an introduction to Schubert calculus see [15], [20], and for connections with combinatorics see [27]. In this

section we explain some recent results of Alexander Postnikov [23] on a quantum deformation of $H^*(\text{Gr}_{kn})$. Further details and references may be found in [23].

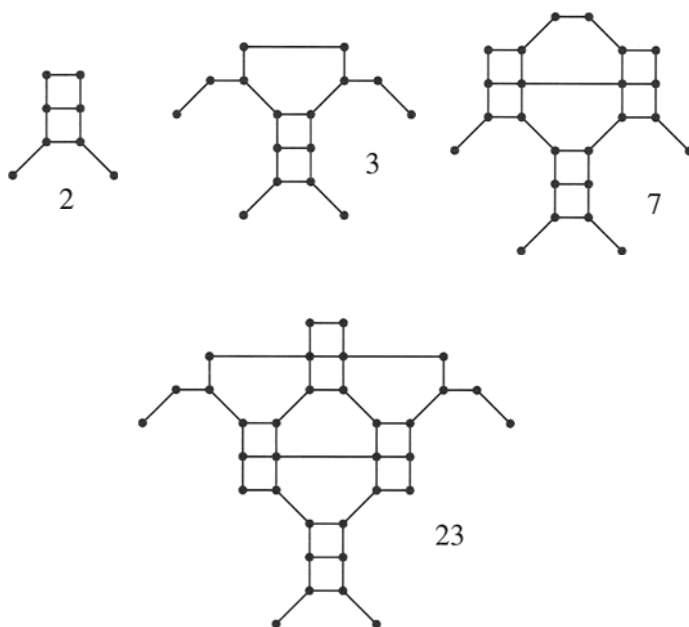


Figure 3. The Somos-4 graphs.

A basis for the cohomology ring $H^*(\text{Gr}_{kn})$ consists of **Schubert classes** σ_λ , where λ ranges over all partitions whose shape fits in a $k \times (n - k)$ rectangle, i.e., $\lambda = (\lambda_1, \dots, \lambda_k)$ where $n - k \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0$. Let P_{kn} denote the set of all such partitions, so

$$\#P_{kn} = \text{rank } H^*(\text{Gr}_{kn}) = \binom{n}{k}.$$

The Schubert classes σ_λ are the cohomology classes of the **Schubert varieties** $\Omega_\lambda \subset \text{Gr}_{kn}$, which are defined by simple geometric conditions, viz., certain bounds on the dimensions of the intersections of a subspace $X \in \text{Gr}_{kn}$ with the subspaces V_i in a fixed flag $\{0\} = V_0 \subset V_1 \subset \dots \subset V_n = \mathbb{C}^n$. Multiplication in the ring $H^*(\text{Gr}_{kn})$ is given by

$$(5) \quad \sigma_\mu \sigma_\nu = \sum_{\lambda \in P_{kn}} c_{\mu\nu}^\lambda \sigma_\lambda,$$

where $c_{\mu\nu}^\lambda$ is a **Littlewood–Richardson coefficient**, described combinatorially by the famous **Littlewood–Richardson rule** (e.g., [31, Appendix A]).

Thus $c_{\mu\nu}^\lambda$ has a geometric interpretation as the intersection number of the Schubert varieties $\Omega_\mu, \Omega_\nu, \Omega_\lambda$. More concretely,

$$(6) \quad c_{\mu\nu}^\lambda = \#(\tilde{\Omega}_\mu \cap \tilde{\Omega}_\nu \cap \tilde{\Omega}_{\lambda^\vee}),$$

the number of points of Gr_{kn} contained in the intersection $\tilde{\Omega}_\mu \cap \tilde{\Omega}_\nu \cap \tilde{\Omega}_{\lambda^\vee}$, where $\tilde{\Omega}_\sigma$ denotes a generic translation of Ω_σ and λ^\vee is the **complementary partition** $(n - k - \lambda_k, \dots, n - k - \lambda_1)$. Equivalently, $c_{\mu\nu}^\lambda$ is the number of k -dimensional subspaces of \mathbb{C}^n satisfying all of the geometric conditions defining $\tilde{\Omega}_{\lambda^\vee}$, $\tilde{\Omega}_\mu$, and $\tilde{\Omega}_\nu$.

The cohomology ring $H^*(\text{Gr}_{kn})$ can be deformed into a “quantum cohomology ring” $\text{QH}^*(\text{Gr}_{kn})$, which specializes to $H^*(\text{Gr}_{kn})$ by setting $q = 0$. More precisely, let Λ_k denote the ring of symmetric polynomials over \mathbb{Z} in the variables x_1, \dots, x_k . Thus

$$\Lambda_k = \mathbb{Z}[e_1, \dots, e_k],$$

where e_i is the i th elementary symmetric function in the variables x_1, \dots, x_k , viz.,

$$e_i = \sum_{1 \leq j_1 < j_2 < \dots < j_i \leq k} x_{j_1} x_{j_2} \cdots x_{j_i},$$

Then we have the ring isomorphism

$$(7) \quad H^*(\text{Gr}_{kn}) \cong \Lambda_k / (h_{n-k+1}, \dots, h_n),$$

where h_i denotes a complete homogeneous symmetric function (the sum of all distinct monomials of degree i in the variables x_1, \dots, x_k). The isomorphism (7) associates the Schubert class $\sigma_\lambda \in H^*(\text{Gr}_{kn})$ with the (image of) the **Schur function** $s_\lambda(x_1, \dots, x_k)$. The Schur functions s_λ , where λ has at most k parts, form a basis for the abelian group Λ_k with many remarkable combinatorial and algebraic properties [21], [31, Ch. 7]. An important property of Schur functions is their **stability**, viz.,

$$s_\lambda(x_1, \dots, x_k, 0) = s_\lambda(x_1, \dots, x_k),$$

which allows us to define the Schur function $s_\lambda = s_\lambda(x_1, x_2, \dots)$ in infinitely many variables x_i by $s_\lambda = \lim_{k \rightarrow \infty} s_\lambda(x_1, \dots, x_k)$. It follows from (5) and (7) that

$$(8) \quad s_\mu s_\nu = \sum_{\lambda} c_{\mu\nu}^\lambda s_\lambda,$$

the “usual” definition of the Littlewood–Richardson coefficients $c_{\mu\nu}^\lambda$.

The quantum cohomology ring $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ differs from $H^*(\mathrm{Gr}_{kn})$ in just one relation: we must enlarge the coefficient ring to $\mathbb{Z}[q]$ and replace the relation $h_n = 0$ with $h_n = (-1)^{k-1}q$. Thus

$$(9) \quad \mathrm{QH}^*(\mathrm{Gr}_{kn}) \cong \Lambda_k \otimes \mathbb{Z}[q]/(h_{n-k+1}, \dots, h_{n-1}, h_n + (-1)^k q).$$

A basis for $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ remains those σ_λ whose shape fits in a $k \times (n-k)$ rectangle, and under the isomorphism (9) σ_λ continues to correspond to the Schur function s_λ . Now, however, the usual multiplication $\sigma_\mu \sigma_\nu$ of Schubert classes has been deformed into a “quantum multiplication” $\sigma_\mu * \sigma_\nu$. It has the form

$$(10) \quad \sigma_\mu * \sigma_\nu = \sum_{d \geq 0} \sum_{\substack{\lambda \vdash |\mu| + |\nu| - dn \\ \lambda \in \mathcal{P}_{kn}}} q^d C_{\mu\nu}^{\lambda, d} \sigma_\lambda,$$

where $C_{\mu\nu}^{\lambda, d} \in \mathbb{Z}$. The geometric significance of the coefficients $C_{\mu\nu}^{\lambda, d}$ (and the motivation for defining $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ in the first place) is that they count the number of rational curves of degree d in Gr_{kn} that meet fixed generic translates of the Schubert varieties Ω_{λ^\vee} , Ω_μ , and Ω_ν . (Naively, a **rational curve** of degree d in Gr_{kn} is a set

$$C = \{(f_1(s, t), f_2(s, t), \dots, f_{\binom{n}{k}}(s, t)) \in P^{\binom{n}{k}-1}(\mathbb{C}) : s, t \in \mathbb{C}\},$$

where

$$f_1(x, y), \dots, f_{\binom{n}{k}}(x, y)$$

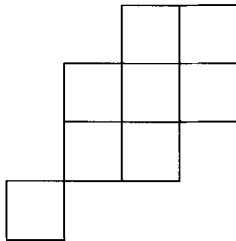
are homogeneous polynomials of degree d such that $C \subset \mathrm{Gr}_{kn}$.) Since a rational curve of degree 0 in Gr_{kn} is just a point of Gr_{kn} , we recover in the case $d = 0$ the geometric interpretation (6) of ordinary Littlewood–Richardson coefficients $c_{\mu\nu}^\lambda = C_{\mu\nu}^{\lambda, 0}$. The numbers $C_{\mu\nu}^{\lambda, d}$ are known as (3-point) **Gromov–Witten invariants**. From this geometric interpretation of Gromov–Witten invariants it follows that $C_{\mu\nu}^{\lambda, d} \geq 0$. No algebraic or combinatorial proof of this inequality is known using equation (9) as the definition of $\mathrm{QH}^*(\mathrm{Gr}_{kn})$, and it is a fundamental open problem to find such a proof.

The primary contribution of Postnikov is a combinatorial description of a new generalization of (skew) Schur functions whose expansion into Schur functions has coefficients which are the Gromov–Witten invariants. This description leads to a better understanding of earlier results as well as a host of new results.

We begin by reviewing the combinatorial definition of skew Schur functions.

Let μ, λ be partitions with $\mu \subseteq \lambda$, i.e., $\mu_i \leq \lambda_i$ for all i . The pair (μ, λ) is called a **skew partition**, often denoted λ/μ . The **diagram** of λ/μ consists of

the diagram of λ with μ removed (from the upper-left corner). For example, the diagram of $(4, 4, 3, 1)/(2, 1, 1)$ is given by



We sometimes call the diagram of λ/μ a **skew shape**.

A **semistandard Young tableau** (SSYT) of shape λ/μ consists of the diagram of λ/μ with a positive integer placed in each square, so that the rows are weakly increasing and columns strictly increasing. If T is an SSYT with a_i occurrences of the entry i , then write

$$(11) \quad x^T = x_1^{a_1} x_2^{a_2} \cdots.$$

Hence the total degree of the monomial x^T is $|\lambda/\mu| = |\lambda| - |\mu|$. Define the **skew Schur function** $s_{\lambda/\mu}$ by

$$s_{\lambda/\mu} = \sum_T x^T,$$

where T ranges over all SSYT of shape λ/μ . The basic facts concerning $s_{\lambda/\mu}$ are the following:

- Let $\mu = \emptyset$, so λ/μ is just the “ordinary” partition λ . Then $s_{\lambda/\emptyset} = s_\lambda$, the “ordinary” Schur function.
- $s_{\lambda/\mu}$ is a symmetric function, whose expansion in terms of Schur functions is given by

$$(12) \quad s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu},$$

where $c_{\mu\nu}^{\lambda}$ denotes a Littlewood–Richardson coefficient (see, e.g., [31, (7.164)]).

We see from equations (8) and (12) that $c_{\mu\nu}^{\lambda}$ has two “adjoint” descriptions, one as a coefficient in a product of Schur functions and one as a coefficient in a skew Schur function. We have already seen in equation (10) the quantum analogue of (8) so now we would like to do the same for (12). We do this by generalizing the definition of a skew shape to a **toric shape**. (Postnikov develops the theory of toric shapes within the more general framework of **cylindric**

shapes, but we will deal directly with toric shapes.) An ordinary skew shape is a certain subset of squares of a $k \times (n - k)$ rectangle. A toric shape is a certain subset τ of squares of a $k \times (n - k)$ torus (which we regard as a $k \times (n - k)$ rectangle with the left and right edges identified, and the top and bottom edges identified). The precise definition follows from the indexing defined below. Figure 4 illustrates a typical toric shape (taken from [23]). (If a row or column forms a loop around the torus, then we must also specify which square is the initial square of the row or column, and these specifications must satisfy a natural consistency condition which we hope Figure 5 makes clear. Henceforth we will ignore the minor modifications needed in our definitions and results when the shape contains toroidal loops.)

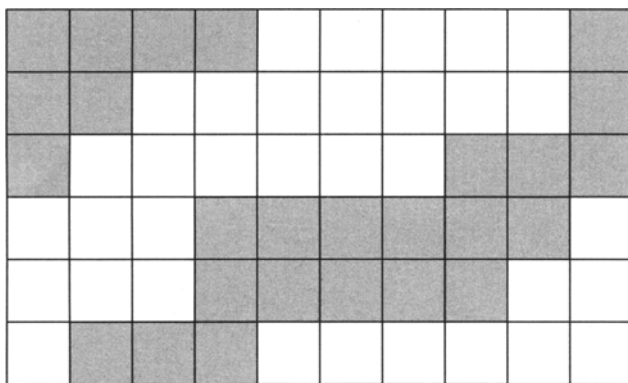


Figure 4. A toric shape in a 6×10 rectangle.

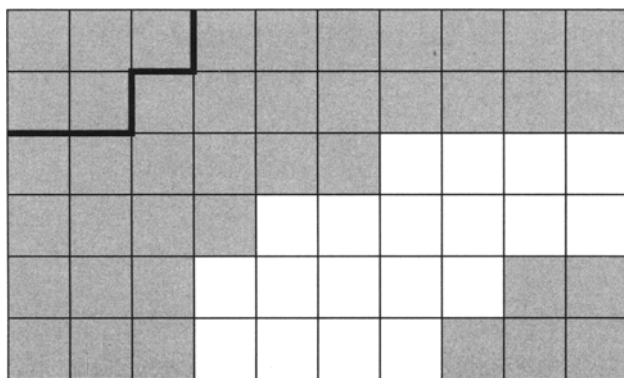


Figure 5. A toric shape with loops.

If τ is a toric shape, then we define a **semistandard toric tableau** (SSTT) of shape τ in exact analogy to the definition of an SSYT: place positive integers into the squares of τ so that every row is weakly increasing and every column strictly increasing. Figure 6 shows an SSTT of the shape τ given by Figure 4.

2	2	4	6						2
3	5								3
4							1	2	4
			1	2	2	2	2	5	
			3	3	4	4	4		
	1	2	4						

Figure 6. A semistandard toric tableau.

We now explain a method of indexing a toric shape τ by a triple $\lambda/d/\mu$, where λ and μ are partitions and $d \geq 0$. We will illustrate this indexing with the toric shape of Figure 4. In Figure 7 we have placed the ordinary shape $\mu = (9, 9, 7, 3, 3, 1)$, outlined with dark solid lines, on the 6×10 torus R . It is the largest shape contained in R whose intersection with τ is an ordinary shape. Similarly, translated $d = 2$ diagonal steps from μ is the shape $\lambda = (9, 7, 6, 2, 2, 0)$, outlined with dark broken lines (and drawn for clarity to extend beyond R but regarded as being on the torus R). It is the largest shape whose upper-left-hand corner is a diagonal translation of the upper-left-hand corner of μ and whose intersection with the complement of τ is a subshape of λ . Thus we rewrite the shape τ as

$$\tau = \lambda/d/\mu = (9, 7, 6, 2, 2, 0)/2/(9, 9, 7, 3, 3, 1).$$

This representation is not unique since any square of R could be taken as the

upper-right corner, but this is irrelevant to the statement of Theorem 3.1 below.

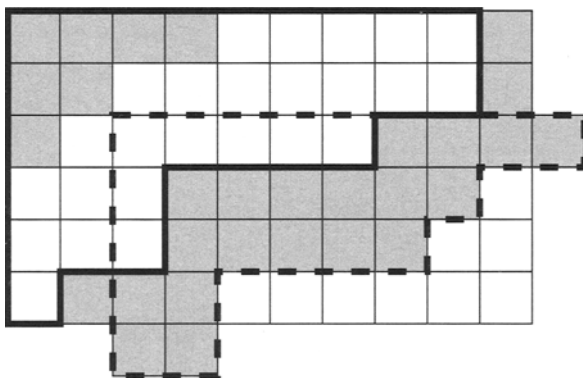


Figure 7. The toric shape $(9, 7, 6, 2, 2, 0)/2/(9, 9, 7, 3, 3, 1)$.

We now define the **toric Schur function** $s_{\lambda/d/\mu}$ exactly in analogy to the skew Schur function $s_{\lambda/\mu} = s_{\lambda/0/\mu}$, viz.,

$$s_{\lambda/d/\mu} = \sum_T x^T,$$

summed over all SSTT of shape $\lambda/d/\mu$, where x^T is defined exactly as in (11). The remarkable main theorem of Postnikov [23, Thm. 6.3] is the following.

3.1 THEOREM: *Let $\lambda/d/\mu$ be a toric shape contained in a $k \times (n - k)$ torus. Then*

$$(13) \quad s_{\lambda/d/\mu}(x_1, \dots, x_k) = \sum_{\nu \in P_{kn}} C_{\mu\nu}^{\lambda,d} s_{\nu}(x_1, \dots, x_k).$$

Note: The above theorem shows in particular that the expansion of the toric Schur function $s_{\lambda/d/\mu}(x_1, \dots, x_k)$ into Schur functions has nonnegative coefficients, i.e., $s_{\lambda/d/\mu}(x_1, \dots, x_k)$ is **Schur positive**. We mentioned above that no “direct” proof using (10) was known that $C_{\mu\nu}^{\lambda,d} \geq 0$. The same is true using (13) as the definition of $C_{\mu\nu}^{\lambda,d}$. Bertram, Ciocan-Fontaine, and Fulton [4] (see also [23, §3]) give a formula for $C_{\mu\nu}^{\lambda,d}$ as an alternating sum of Littlewood–Richardson coefficients, but again no direct proof of positivity is known. On the other hand, if we take $s_{\lambda/d/\mu}$ in more than k variables, then it is still true (and not difficult to prove) that $s_{\lambda/d/\mu}$ remains a symmetric function, but it need not be Schur positive.

Theorem 3.1 was used by Postnikov to obtain many properties of Gromov–Witten invariants, some already known and some new. For example, he gives a transparent explanation of a “hidden” symmetry of Gromov–Witten invariants, and he solves the previously open problem of describing which powers q^d appear with nonzero coefficients in a quantum product $\sigma_\lambda * \sigma_\mu$ of quantum Schubert classes. This latter problem is equivalent to determining which $\mu^\vee/d/\lambda$ form a valid toric shape, where μ^\vee is the complementary partition

$$(n - k - \mu_k, \dots, n - k - \mu_1).$$

For further details, see [23].

4. Toric h -vectors and intersection cohomology

A **convex polytope** is the convex hull of finitely many points in a Euclidean space. The subject of convex polytopes has undergone spectacular progress in recent years. New tools from commutative algebra, exterior algebra, algebraic geometry, and other fields have led to solutions of previously intractable problems. Convex polytopes have also arisen in unexpected new areas and have been applied to problems in these areas. Computer science has raised a host of new problems related to convex polytopes and has greatly increased our ability to handle specific examples. A good general reference is [35]. We will discuss a recent breakthrough related to combinatorial properties of convex polytopes, followed by a brief description of a recent result on matroid complexes. Both results have a similar “hard Lefschetz” flavor.

Let \mathcal{P} be a d -dimensional (convex) polytope, or d -polytope for short. Let $f_i = f_i(\mathcal{P})$ denote the number of i -dimensional faces of \mathcal{P} , where we set $f_{-1} = 1$ (regarding the empty face as having dimension -1). The vector $f(\mathcal{P}) = (f_0, f_1, \dots, f_{d-1})$ is called the **f-vector** of \mathcal{P} . The general problem of characterizing f -vectors of polytopes seem hopeless, but when \mathcal{P} is **simplicial** (i.e., every proper face of \mathcal{P} is a simplex) then a complete characterization is known (the **g-theorem** for simplicial polytopes). Define the **h-vector** $h(\mathcal{P}) = (h_0, h_1, \dots, h_d)$ of a simplicial polytope \mathcal{P} with f -vector (f_0, \dots, f_{d-1}) by

$$(14) \quad \sum_{i=0}^d f_{i-1}(x-1)^{d-i} = \sum_{i=0}^d h_i x^{d-i}.$$

The h -vector and f -vector convey the same information. A vector $(h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$ is the h -vector of a simplicial d -polytope if and only if the following conditions are satisfied:

- (G₁) $h_0 = 1$.
- (G₂) $h_i = h_{d-i}$ for $0 \leq i \leq d$ (the Dehn–Sommerville equations).
- (G₃) $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$ (the Generalized Lower Bound Conjecture, or GLBC).
- (G₄) Certain non-polynomial inequalities (which we call the **g-inequalities**) asserting that the differences $g_i := h_i - h_{i-1}$ cannot grow too rapidly for $1 \leq i \leq \lfloor d/2 \rfloor$. We will not state these conditions here.

The sufficiency of the above conditions was proved by Billera and Lee, and the necessity by Stanley. A brief exposition of this result, with further references, appears in [29, §3.1]. The basic idea of the proof of necessity is the following. First, condition G₁ is trivial, and G₂ is well-known and not difficult to prove. To prove G₃ and G₄, slightly perturb the vertices of \mathcal{P} so that they have rational coordinates. Since \mathcal{P} is simplicial, small perturbations of the vertices do not change the combinatorial type, and hence leave the f -vector and h -vector invariant. Once \mathcal{P} has rational vertices, we can construct a projective algebraic variety $X_{\mathcal{P}}$, the **toric variety** corresponding to the normal fan $\Sigma_{\mathcal{P}}$ of \mathcal{P} , whose cohomology ring (say over \mathbb{R}) has the form

$$H^*(X_{\mathcal{P}}; \mathbb{R}) = H^0(X_{\mathcal{P}}; \mathbb{R}) \oplus H^2(X_{\mathcal{P}}; \mathbb{R}) \oplus \cdots \oplus H^{2d}(X_{\mathcal{P}}; \mathbb{R}),$$

where $\dim H^{2i}(X_{\mathcal{P}}; \mathbb{R}) = h_i(\mathcal{P})$. (The normal fan $\Sigma_{\mathcal{P}}$ is defined below for any polytope \mathcal{P} .) Write for short $H^i = H^i(X_{\mathcal{P}}; \mathbb{R})$. Let Y be a generic hyperplane section of $X_{\mathcal{P}}$, with corresponding cohomology class $\omega = [Y] \in H^2$. The variety $X_{\mathcal{P}}$ has sufficiently nice singularities (finite quotient singularities) so that the hard Lefschetz theorem holds. This means that if $i < d/2$, then the map $\omega^{d-2i}: H^{2i} \rightarrow H^{2(d-i)}$ given by $u \mapsto \omega^{d-2i}u$ is a bijection. In particular, this implies that if $i < d/2$ then the map $\omega: H^{2i} \rightarrow H^{2(i+1)}$ is injective. Hence $\dim H^{2i} \leq \dim H^{2(i+1)}$ for $i < \lfloor d/2 \rfloor$, so G₃ follows. To obtain G₄, we use the fact that $H^*(X_{\mathcal{P}}; \mathbb{R})$ is a graded \mathbb{R} -algebra generated by H^2 ; details may be found in the reference cited above.

What happens when we try to extend this reasoning to nonsimplicial polytopes \mathcal{P} ? We can still define the variety $X_{\mathcal{P}}$, but unless \mathcal{P} is **rational** (i.e., has rational vertices) $X_{\mathcal{P}}$ will not have finite type (so will not be what is normally meant by an algebraic variety) and little can be said. Thus for now assume that \mathcal{P} has rational vertices. Unfortunately $X_{\mathcal{P}}$ now has more complicated singularities, and the cohomology ring $H^*(X_{\mathcal{P}}; \mathbb{R})$ is poorly behaved. In particular, the Betti numbers $\dim H^i(X_{\mathcal{P}}; \mathbb{R})$ depend on the embedding of \mathcal{P} into the ambient Euclidean space. However, the theory of **intersection cohomology** intro-

duced by Goresky and MacPherson [17], [18] in 1980 yields “nice” cohomology for singular spaces.

In particular, if \mathcal{P} is any rational d -polytope then the intersection cohomology of the toric variety $X_{\mathcal{P}}$ has the form

$$\mathrm{IH}(X_{\mathcal{P}}; \mathbb{R}) = \mathrm{IH}^0(X_{\mathcal{P}}; \mathbb{R}) \oplus \mathrm{IH}^2(X_{\mathcal{P}}; \mathbb{R}) \oplus \cdots \oplus \mathrm{IH}^{2d}(X_{\mathcal{P}}; \mathbb{R}),$$

where each $\mathrm{IH}^{2i}(X_{\mathcal{P}}; \mathbb{R})$ is a finite-dimensional real vector space whose dimension h_i depends only on the combinatorial type of \mathcal{P} and not on its embedding into Euclidean space. The vector

$$h(\mathcal{P}) = (h_0, h_1, \dots, h_d)$$

is called the **toric h -vector** (or **generalized h -vector**) of \mathcal{P} . If \mathcal{P} is simplicial, then intersection cohomology and singular cohomology coincide, so the toric h -vector coincides with the ordinary h -vector.

The combinatorial description of the toric h -vector is quite subtle. For any polytope \mathcal{P} , define two polynomials $f(\mathcal{P}, x)$ and $g(\mathcal{P}, x)$ recursively as follows. If $\mathcal{P} = \emptyset$ then $f(\emptyset, x) = g(\emptyset, x) = 1$. If $\dim \mathcal{P} \geq 0$ then define

$$f(\mathcal{P}, x) = \sum_{\mathcal{Q}} g(\mathcal{Q}, x)(x-1)^{\dim \mathcal{P} - \dim \mathcal{Q} - 1},$$

where \mathcal{Q} ranges over all faces (including \emptyset) of \mathcal{P} except $\mathcal{Q} = \mathcal{P}$. Finally, if $\dim \mathcal{P} = d \geq 0$ and $f(\mathcal{P}, x) = h_0 + h_1x + \cdots$, then define

$$g(\mathcal{P}, x) = h_0 + (h_1 - h_0)x + (h_2 - h_1)x^2 + \cdots + (h_m - h_{m-1})x^m,$$

where $m = \lfloor d/2 \rfloor$. It is easy to see that we have defined $f(\mathcal{P}, x)$ and $g(\mathcal{P}, x)$ recursively for all polytopes \mathcal{P} . For instance, let σ_j denote a j -dimensional simplex and \mathcal{C}_j a j -dimensional cube. Suppose we have computed that $g(\sigma_0) = g(\sigma_1) = 1$ and $g(\mathcal{C}_2, x) = 1 + x$. Then

$$\begin{aligned} f(\mathcal{C}_3, x) &= 6(x+1) + 12(x-1) + 8(x-1)^2 + (x-1)^3 \\ &= x^3 + 5x^2 + 5x + 1, \end{aligned}$$

and $g(\mathcal{C}_3, x) = 1 + 4x$.

It is easy to see that $\deg f(\mathcal{P}, x) = d$, so $f(\mathcal{P}, x) = h_0 + h_1x + \cdots + h_dx^d$. Then $h(\mathcal{P}) = (h_0, h_1, \dots, h_d)$ is the toric h -vector of \mathcal{P} when \mathcal{P} is rational, and we can define the toric h -vector of any polytope \mathcal{P} in this manner. As mentioned above, it coincides with the usual h -vector when \mathcal{P} is simplicial. It is trivial that

$h_0 = 1$, and it was first shown in [28, Thm. 2.4] that $h_i = h_{d-i}$ for all i (the generalized Dehn–Sommerville equations). Since $h_i = \dim \mathrm{IH}^{2i}(X_{\mathcal{P}}; \mathbb{R})$ when \mathcal{P} is rational, we have $h_i \geq 0$ in this case. Moreover, $\mathrm{IH}(X_{\mathcal{P}}; \mathbb{R})$ is a (graded) module over $H^*(X_{\mathcal{P}}; \mathbb{R})$, and the hard Lefschetz theorem continues to hold as follows: the map $\omega: \mathrm{IH}^{2i} \rightarrow \mathrm{IH}^{2(i+1)}$ defined by $u \mapsto \omega u$, where $\omega \in H^2$ as above, is injective for $i < d/2$. Hence as before we get $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$. Thus conditions G_1 , G_2 , and G_3 of the g -theorem continue to hold for any rational polytope (using the toric h -vector). However, intersection cohomology does not have a ring structure, and it remains open whether G_4 holds for all rational polytopes.

Not all polytopes are rational, i.e., there exist polytopes \mathcal{P} for which any polytope in \mathbb{R}^n with the same combinatorial type as \mathcal{P} cannot have only rational vertices [35, Exam. 6.21]. It was conjectured in [28] that conditions G_3 – G_4 hold for all convex polytopes. An approach toward proving G_3 would be to construct a “nice” analogue of the toric variety $X_{\mathcal{P}}$ when \mathcal{P} is nonrational. It doesn’t seem feasible to do this. Instead, we can try to construct an analogue of the intersection cohomology $\mathrm{IH}(X_{\mathcal{P}}; \mathbb{R})$ when \mathcal{P} is nonrational, i.e., a graded \mathbb{R} -algebra $\mathrm{IH}(\mathcal{P}) = \mathrm{IH}^0(\mathcal{P}) \oplus \mathrm{IH}^2(\mathcal{P}) \oplus \cdots \oplus \mathrm{IH}^{2d}(\mathcal{P})$ that becomes $\mathrm{IH}(X_{\mathcal{P}}; \mathbb{R})$ when \mathcal{P} is rational and which satisfies the two conditions:

$$(P_1) \quad \dim \mathrm{IH}^{2i}(\mathcal{P}) = h_i(\mathcal{P}),$$

$$(P_2) \quad \mathrm{IH}(\mathcal{P}) \text{ is a module over some ring containing an element } l \text{ of degree 2 that satisfies the conditions of hard Lefschetz theorem, i.e., for } i < d/2 \text{ the map } l^{d-2i}: \mathrm{IH}^{2i}(\mathcal{P}) \rightarrow \mathrm{IH}^{2(d-i)}(\mathcal{P}) \text{ is a bijection.}$$

The first step in the above program was the definition of $\mathrm{IH}(\mathcal{P})$ due to Barthel, Brassel, Fieseler and Kaup [1] and to Bressler and Lunts [6]. The precise definition is rather technical; we include it here (following [19]) so that even readers without the necessary background will have some idea of its flavor. With the polytope $\mathcal{P} \subset \mathbb{R}^n$ we can associate the **normal fan** $\Sigma = \Sigma_{\mathcal{P}}$, i.e., the set of all cones of linear functions which are maximal on a fixed face of \mathcal{P} (e.g., [35, Exam. 7.3]). In other words, if F is a nonempty face of \mathcal{P} then define

$$N_F := \{f \in (\mathbb{R}^n)^*: F \subseteq \{x \in \mathcal{P}: f(x) = \max_{y \in \mathcal{P}} f(y)\}\}.$$

The set of all cones N_F forms the normal fan of \mathcal{P} ; it is a **complete fan** in $(\mathbb{R}^n)^*$, i.e., any two cones intersect in a common face of both, and the union of all N_F ’s is $(\mathbb{R}^n)^*$. Define a sheaf \mathcal{A}_{Σ} , the **structure sheaf** of Σ , as follows. For each cone $\sigma \in \Sigma$ define the stalk $\mathcal{A}_{\Sigma, \sigma} = \mathrm{Sym}(\mathrm{span} \sigma)^*$, the space of polynomial functions on σ . The restriction map $\mathcal{A}_{\Sigma, \sigma} \rightarrow \mathcal{A}_{\Sigma}(\partial \sigma)$ is defined by restriction of

functions (where ∂ denotes boundary). Thus \mathcal{A}_Σ is a sheaf of algebras, naturally graded by degree (where we define conewise linear functions to have degree 2), so

$$A_\Sigma = A_\Sigma^0 \oplus A_\Sigma^2 \oplus A_\Sigma^4 \oplus \cdots.$$

Let $A = \text{Sym}(\mathbb{R}^n)^*$ denote the space of polynomial functions on all of \mathbb{R}^n . Multiplication with elements of A gives \mathcal{A}_Σ the structure of a sheaf of A -modules.

An **equivariant intersection cohomology sheaf** \mathcal{L}_Σ of Σ is a sheaf of \mathcal{A}_Σ -modules satisfying the following three properties:

- (E₁) (normalization) $\mathcal{L}_{\Sigma,0} = \mathbb{R}$.
- (E₂) (local freeness) $\mathcal{L}_{\Sigma,\sigma}$ is a free $\mathcal{A}_{\Sigma,\sigma}$ -module for any $\sigma \in \Sigma$.
- (E₃) (minimal flabbiness) Let I be the ideal of A generated by homogeneous linear functions, and for any A -module M write $\overline{M} = M/IM$. Then modulo the ideal I the restriction map induces an isomorphism

$$\overline{\mathcal{L}_{\Sigma,\sigma}} \rightarrow \overline{\mathcal{L}_\Sigma(\partial\sigma)}.$$

Equivariant intersection cohomology sheaves exist for any fan Σ , and any two of them are isomorphic. Hence we may call a sheaf of \mathcal{A}_Σ -modules satisfying E₁, E₂, and E₃ *the* equivariant intersection homology sheaf. We have $\mathcal{L}_\Sigma \simeq \mathcal{A}_\Sigma$ if and only if the fan Σ is simplicial, in which case it coincides with the usual equivariant cohomology sheaf. (A good introduction to equivariant cohomology can be found in [8].) The (non-equivariant) **intersection cohomology** of the fan Σ is defined to be the \mathcal{A} -module of global sections of the intersection cohomology sheaf modulo the ideal I :

$$\text{IH}(\Sigma) = \overline{\mathcal{L}(\Sigma)}.$$

Since $\mathcal{L}(\Sigma)$ is a graded A -module and I is a graded ideal, $\text{IH}(\Sigma)$ inherits a natural grading:

$$\text{IH}(\Sigma) = \text{IH}(\Sigma)^0 \oplus \text{IH}(\Sigma)^2 \oplus \cdots.$$

By definition \mathcal{A}_Σ^2 (the degree two part of \mathcal{A}_Σ) consists of conewise linear functions on Σ . The restriction l_σ of l to σ is linear on σ and hence extends to a unique linear function $l_\sigma \in A^2$. A function $l \in \mathcal{A}_\Sigma^2$ is called **strictly convex** if $l_\sigma(v) < l(v)$ for any maximal cone $\sigma \in \Sigma$ and any $v \notin \sigma$. A complete fan Σ is called **projective** if there exists a strictly convex function $l \in \mathcal{A}_\Sigma^2$.

Now let \mathcal{P} be any convex polytope, and let $\Sigma_{\mathcal{P}}$ denote the normal fan of \mathcal{P} . The normal fan $\Sigma_{\mathcal{P}}$ is easily seen to be projective. Write $\text{IH}(\mathcal{P})$ for $\text{IH}(\Sigma_{\mathcal{P}})$.

Bressler and Lunts [6] established a number of fundamental properties of the intersection cohomology $\mathrm{IH}(\mathcal{P})$ of a convex polytope \mathcal{P} . They showed that if $\dim \mathcal{P} = d$ then the grading of $\mathrm{IH}(\mathcal{P})$ has the form

$$\mathrm{IH}(\mathcal{P}) = \mathrm{IH}^0(\mathcal{P}) \oplus \mathrm{IH}^2(\mathcal{P}) \oplus \cdots \oplus \mathrm{IH}^{2d}(\mathcal{P}),$$

where each $\mathrm{IH}^{2i}(\mathcal{P})$ is a finite-dimensional vector space satisfying Poincaré duality, so $\mathrm{IH}^{2i}(\mathcal{P}) \simeq \mathrm{IH}^{2(d-i)}(\mathcal{P})$. They conjectured that $\mathrm{IH}(\mathcal{P})$ has the hard Lefschetz property, i.e., if $l \in \mathcal{A}_\Sigma^2$ is strictly convex, then for $i < d/2$ the map

$$l^{d-i}: \mathrm{IH}^{2i}(\mathcal{P}) \rightarrow \mathrm{IH}^{2(d-i)}(\mathcal{P})$$

is a bijection. They showed that if $\mathrm{IH}(\mathcal{P})$ does have the hard Lefschetz property, then $\dim \mathrm{IH}^{2i}(\mathcal{P}) = h_i(\mathcal{P})$. Hence, as explained above, the toric h -vector of any polytope would satisfy property G_3 (the GLBC).

The conjecture of Bressler and Lunts, that $\mathrm{IH}(\mathcal{P})$ has the hard Lefschetz property, was first proved by Kalle Karu [19]. Karu actually proves a stronger result, the **Hodge–Riemann–Minkowski bilinear relations**. To state this result, the Poincaré duality on intersection cohomology gives a pairing

$$\mathrm{IH}^{d-i}(\mathcal{P}) \times \mathrm{IH}^{d+i}(\mathcal{P}) \rightarrow \mathbb{R},$$

denoted $\langle x, y \rangle$. If $l \in \mathcal{A}_\Sigma^2$ is strictly convex, then we define a quadratic form Q_l on $\mathrm{IH}^{d-i}(\mathcal{P})$ by $Q_l(x) = \langle l^i x, x \rangle$. The **primitive intersection cohomology** $\mathrm{IP}^{d-i}(\mathcal{P})$ is defined to be the kernel of the map

$$l^{i+1}: \mathrm{IH}^{d-i}(\mathcal{P}) \rightarrow \mathrm{IH}^{d+i+2}(\mathcal{P}).$$

The Hodge–Riemann–Minkowski bilinear relations then state that the quadratic form $(-1)^{(d-i)/2} Q_l$ is positive definite on $\mathrm{IP}^{d-i}(\mathcal{P})$ for all $i \geq 0$. (Note that $\mathrm{IP}^{d-i}(\mathcal{P}) = 0$, or even $\mathrm{IH}^{d-i}(\mathcal{P}) = 0$, unless $(d-i)/2$ is an integer.) These relations were proved for simplicial polytopes by McMullen [22]. A simpler proof was later given by Timorin [34]. Very roughly, the idea of the proof of Karu is: give a suitable simplicial subdivision Δ of the fan $\Sigma_{\mathcal{P}}$ and “lift” the Hodge–Riemann–Minkowski relations from Δ to Σ .

The proof of Karu has the defect that the Poincaré pairing $\langle \cdot, \cdot \rangle$ depends on the choice of the subdivision Δ and on the embedding $\mathcal{L}(\Sigma) \subset \mathcal{L}(\Delta)$. In the words of Bressler and Lunts [7], “this ambiguity makes the proof unnecessarily heavy and hard to follow.” Bressler and Lunts define in this same paper a *canonical* pairing that considerably simplifies the proof of Karu. They also show

that Karu's pairing is independent of the choices made to define it and in fact coincides with the canonical pairing. Finally Barthel, Brasselet, Fieseler and Kaup [2] give a "direct" approach to the proof of Bressler and Lunts, replacing derived categories with elementary sheaf theory and commutative algebra.

We conclude this section by briefly mentioning a further recent result concerning h -vectors and a variation of the hard Lefschetz theorem. Undefined terminology may be found, e.g., in [29]. A **matroid complex** is a (finite) abstract simplicial complex Δ on a vertex set V such that the restriction of Δ to any subset of V is pure, i.e., all maximal faces have the same dimension. The f -vector $f(\Delta) = (f_0, f_1, \dots, f_{d-1})$ is defined just as for polytopes (where $\dim \Delta = d - 1$), and the h -vector $h(\Delta) = (h_0, h_1, \dots, h_d)$ is defined by equation (14). It is well known that matroid complexes are Cohen-Macaulay, so in particular $h_i \geq 0$. It is of considerable interest to obtain further conditions on h -vectors of matroid complexes, and there are many results and conjectures in this direction [29, §III.3]. In particular, Chari [9, Cor. 4, part 2] has shown that if $h_s \neq 0$ and $h_{s+1} = h_{s+2} = \dots = h_d = 0$, then $h_0 \leq h_1 \leq \dots \leq h_{\lfloor s/2 \rfloor}$ and $h_i \leq h_{s-i}$ for $0 \leq i \leq \lfloor s/2 \rfloor$. Because matroid complexes are Cohen-Macaulay, there is a natural graded \mathbb{R} -algebra $A = A_0 \oplus A_1 \oplus \dots \oplus A_s$ such that $\dim A_i = h_i$ (namely, A is the face ring of Δ modulo a linear system of parameters). This suggests that a "half hard Lefschetz theorem" might hold for matroid complexes, viz., there is an element $l \in A_1$ such that the map $l^{d-2i}: A_i \rightarrow A_{d-i}$ is *injective* (rather than bijective) for $0 \leq i \leq \lfloor s/2 \rfloor$. In particular, this would imply the g -inequalities G_4 for the sequence $h_0, h_1, \dots, h_{\lfloor s/2 \rfloor}$. This half hard Lefschetz theorem (for a generic linear system of parameters) was proved by Ed Swartz [33].

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